Simplicity and the Lascar group

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This paper contains a series of easy constructions and observations relating to the Lascar group and to simple theories. [1]

In §1 we review basic model theoretic ideas, relating mostly to model completion and saturated models. We do so in order to introduce a framework very slightly more general than the usual first-order one that will be useful to us, and that we hope may be useful in the future. We will refer to this as "Robinson theories".

In §2 we give an account of Lascar's beautiful construction, associating a compact topological group to first-order theories. Our description, influenced by work of Kim-Pillay, applies to all first order theories. For "G-compact" theories the results coincides with the full Lascar group; for others, if there are any others, it gives a quotient of the full Lascar group. We present the Lascar group as an automorphism group of a compact topological structure associated naturally with the theory, that we call the Kim-Pillay space.

In §3 we find a connection between the Lascar group and certain spaces of theories. In particular, we see that a necessary condition for the existence of a theory with connected Lascar group (in a certain class of theories closed under interpretations), is that there exist a continuous path in the space of theories (within the given class), interpolating between the theory of the empty graph to the theory of the complete graph. It is worth noting that even if we start with a first order theory, this analysis necessarily involves Robinson theories; it would not have been possible without the extension of the framework in §1.

The Lascar group was brought into prominence in recent work of Kim and Pillay, on simple theories. We will discuss simplicity briefly in the introduction to §4. This property was introduced in [Sh1] as a generalization of stability. After a decade of neglect, a few years of intense activity by a number of workers, sparked by Kim's thesis and by some work in finite rank, brought the state of knowledge to nearly the same level as for stable theories. It was found that a theory of independence can be developed that is as coherent and satisfactory as in the stable context, though necessarily with some different features. However, the general theory as described by Kim

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and Pillay used Lascar types rather than ordinary types or strong types.

We show in §4.1 that there exists a rank one simple Robinson theory with connected Lascar group; indeed any compact group can occur in this way. We view this very easy observation as showing that (at least in the more general framework) the Kim-Pillay theory is perfect as it stands; far from a temporary expedient, the use of the Lascar group is fundamental. As suggested by Kim and Pillay, infinitely-definable objects, and in particular equivalence relations, take the role of definable objects in stability theory; and arbitrary compact groups replace the profinite groups of stability theory.

In §4.2-§4.3 we modify an old construction method of stable theories to construct a rank 1, simple, $\aleph_0$-categorical theory. This contrasts with the stable case, where Zil'ber's classification theorem shows that all rank 1 $\aleph_0$-categorical structures are small modifications of projective geometries over finite fields.

The independent amalgamation property characteristic of simple theories has been called, in other contexts, the $P(3)^-$-amalgamation property by Shelah. (This means that the natural indexing set for the substructures involved is the set of proper subsets of a three-element set.) From this point of view simplicity is only one of a family of increasingly stronger conditions, the $P(n)^-$-amalgamation properties. (All these existence properties follow from uniqueness in the stable case.) The cases $n \leq 3$ are the only one with direct implications to *ranks*, or to the structure of definable *sets*, rather than relations, and as such it fits better with more familiar conditions, such as stability. (Most stability theorists are at first surprised to find that the generic pyramid-free hypergraph is simple, while the generic triangle-free graph is not.) In the appendix §5 (text of a letter to Shelah from some years ago), we use this observation to answer a question from [Sh1]: because of the different amalgamation properties, it is consistent with ZFC that not all simple unstable theories have the same saturation spectrum.

1 Quantifier Separation

We wish to generalize somewhat the usual ("first-order") context of model theory. Our original motivation arose from stability theory. If H is a definable subset of a stable structure, then H, with the induced structure, is itself stable. However if H is defined by an infinite set of formulas, the full first-order theory of H is usually unstable; stability theorists often had to use the awkward hypothesis "let H be infinitely-definable in a stable theory", knowing however that the full first order theory can be irrelevant, and standard stability theory applies full well within H.

It turns out that one can define a context, very slightly more general than first-order, in which the main techniques of model theory remain valid (though of course individual results need no longer hold.) We have in mind omitting types, indiscernibles, prime models, compactness, saturated models and saturated Galois theory, stability, ω-stability. Roughly speaking, for countable languages, one takes a (very) small step into $L_{\omega_1,\omega}$, permitting a sentences of the form $(\exists x)\phi \equiv \bigwedge_n \psi_n$. We will give a description along slightly different lines however.

Definition 1 *Let L be a language, Δ a collection of formulas, closed under subformulas and under Boolean combinations. Let Σ,Π denote the set of formulas of the form $(\exists x_1 \ldots x_k)\phi$ (resp. $(\forall x_1 \ldots x_k)\phi$) with $\phi \in \Delta$. We will refer to Δ,Σ,Π as the* basic,existential,universal *formulas, respectively. A* formula *with no further designation will mean a Δ-formula.*
Let T be a universal theory, i.e. a set of sentences from Π. Assume that T decides at least the basic sentences, i.e. if $\phi \in \Delta$ and ϕ is a sentence then $\phi \in T$ or $\neg\phi \in T$. We will say that T is a Robinson theory*, or that T admits* quantifier separation*, if for any two basic formulas $\phi(x, y)$ and $\psi(y, z)$, (where x and z may be taken to be single variables, but y is a tuple of any length), if $T \models \phi(x, y) \Rightarrow \psi(y, z)$ then there exists a $\theta(y) \in \Delta$ such that $T \models \phi(x, y) \Rightarrow \theta(y) \Rightarrow \psi(y, z)$.*

Remarks

1. *Quantifier Interpolation* may seem a more suitable name, but see 1.5.

2. An example is the first order case, where Δ is the set of all formulas. In this case the interpolation property holds with, say, $\theta = (\exists x)\phi$.

3. We have implicitly in mind that a Robinson theory T carries with it the following completion $\tilde{T}$:

$$(\exists x)\phi(x, y) \leftrightarrow \bigwedge C(\phi)$$

where

$$C(\phi) = \{\neg\theta(y) : T \vdash (\forall x)(\forall y)(\theta \Rightarrow \neg\phi)\}$$

$\tilde{T}$ is first-order iff each $C(\phi)$ is finitely generated iff T has a model completion. (And in this case, $\tilde{T}$ is the model completion .) We will refer to this as the *first order case* and say "T is first order", though we really mean $\tilde{T}$.

4. Note that quantifier separation is a property of the universal theory itself, and not of a potential extension. Thus if T is a universal theory in L, with a model completion $\tilde{T}$, (so that the universal part of $\tilde{T}$ is T), then T is Robinson.

5. ("Morley-zation") When working in a general context, we may always take Δ to be the set of quantifier-free formulas (we can achieve this in a definitional expansion of the language, in which every Δ-formula becomes equivalent to a new basic relation.) This is not always convenient however in more concrete situations where specific representations of the language may be available. We will be able to make this assumption throught most of the paper, however; thus if Δ is not otherwise specified, it will be the set of quantifier - free formulas; we will refer to Π as the universal formulas, etc.

6. Another justificiation for the name "Robinson theories" (that I did not see in advance) is that such universal theories have a unique forcing extension in the sense of model theoretic forcing. However, this canonical completion is more easily described in other ways, and is misleading: the idea here is precisely that the higher quantified formulas need *not* be considered.

7. Here and in the sequel, all formulas are taken to be in Δ unless explicitly mentioned otherwise; and $Th(U)$ stands for the set of universal sentences true in U.

We will now give some equivalent forms of the definitions, and verify that some standard model theoretic constructions remain valid. There is room for looking at the validity of failure of deeper results in this context, but we make not attempt here to do so.
We leave somewhat open the word "small substructure". We have in mind that every substructure of interest to us is small in this sense. Possible specific meanings include "countable", "smaller than κ" "set - sized", etc.

Definition 2

- *Let L be a language. A set D defined in an L-structure M using a basic formula of L will be called basic; if parameters from $A \subset M$ are used, we will call D $A-basic$.*
- *An L-structure M is (Δ -) homogeneous if whenever A,B are small substructures of M, and $f : A \to B$ is an isomorphism preserving Δ-formulas, then f extends to an automorphism of M.*
- *An L-structure M is compact if whenever Y is a small collection of M- basic subsets of M, and every finite subcollection has nonempty intersection, then Y has nonempty intersection.*
- *M is a universal domain for L if M is compact and homogeneous. If in addition $Th(M) = T$, we say that M is a universal domain for T.*

Lemma 1.1 *Any universal domain for T is existentially closed among models of T.*

Proof Let $\phi(x, b)$ be a formula over a universal domain $\mathtt{U}$, satisfied in some model N of T. Then N embeds into $\mathtt{U}$, so there exists $b' \in \mathtt{U}$, with the same basic type as b, such that $\phi(x, b')$ has a solution in $\mathtt{U}$. By homogeneity, so does $\phi(x, b)$.

Lemma 1.2 *Let U be a universal domain. Then every Σ_1 set in U is the intersection of a small family of basic sets. Conversely, if U is compact and the above condition holds, then $Th(U)$ is Robinson.*

Proof Let X be a Σ_1 set, defined say over the finite set A. Let F be the collection of all A-basic sets disjoint from X. For any element a, let $T(a) = \{\sigma(a) : \sigma \in Aut(U/A)\}$. Then $T(a) = \cap\{Y : Y \ A \text{-basic}, a \in Y\}$. If $a \notin X$, then $T(a) \cap X = \emptyset$; so by compactness there exists an A-basic Y, $a \in Y$, $Y \in F$. Thus the complement of X is contained in the union of F; so X is the intersection of the complements of the sets in F.

For the converse, we show the separation property. If X_1,X_2 are disjoint Σ_1 sets, then $X_1 = \cap F_1$ and $X_2 = \cap F_2$; by compactness for some $Y_i \in F_i$, $Y_1 \cap Y_2 =$, so Y_1 separates X_1 and X_2.

Lemma 1.3 *If "small" means "$< \kappa$",then any two universal domains for T of cardinality κ are isomorphic*

Proof Standard proof of uniqueness of saturated models; back - and -forth of Δ-isomorphisms.

Proposition 1.1 *Let T be a universal theory, complete for Δ-sentences. T is Robinson iff there exists a universal domain for T (equivalently, iff every model of T embeds into a universal domain.)*

Proof Let $\mathtt{U}$ be a universal domain for T. Assume $T \vdash \phi(x,y) \Rightarrow \psi(y,z)$. We saw that $(\exists y)\phi(x,y)$ is equivalent there to a conjunction of basic formulas F. So $\mathtt{U} \models F(y) \Rightarrow \psi(y,z)$. By compactness, for some $\theta \in F$, $\mathtt{U} \models \theta(y) \Rightarrow \psi(y,z)$.

The converse direction is proved by the standard proof of existence of saturated models.

Notation 1.4 $\mathtt{U}$ *is a universal domain. A,B etc. denote small subsets of U. A subset P of X is of class Σ_1 if there exists a basic subset T of $X \times Y$ for some Y, such that $P = \pi_X(T)$.*

Lemma 1.5 *T is Robinson iff whenever X,Y are disjoint Σ_1 sets, there exists a basic Z separating X and Y, i.e. $X \subset Z$ and $Y \cap Z = \emptyset$. In particular, if X and its complement are Σ_1, then X is basic.*

Lemma 1.6 *Let U be a universal domain, X a U-basic set.*

1. *Let F be a small collection of existential formulas over U. If every finite subset of F has a solution, so does F as a whole.*
2. *Let Y be a small collection of Σ_1 subsets of X, with the finite intersection property. Then Y has nonempty intersection.*
3. *Let X be a U-basic set, and suppose X is invariant under $Aut(U/A)$. Then X is an A-basic set.*

Proof (1) By treating the existentially quantified variables as new free variables, we may assume here that F is a small collection of quantifier-free formulas. We propose to inductively replace the free variables of F by elements of U, in such a way that the system remains finitely satisfiable. Thus let x be one of the free variables. Let F_x be the set of all basic formulas $\phi(x)$ that follow from $Th(U) \cup F$. Then by the compactness assumption, F_x is satisfied by some $a \in U$. By quantifier separation, if $\psi(y,x)$ follows from F then $\psi(y,a)$ is satisfiable: otherwise, by homogeneity, whenever a' realizes the basic type p of a, there is no y with $\psi(y,a')$. So $p \cup \psi(y,x)$

is not satisfied in U, hence by compactness $p_0 \cup \psi(y, x)$ is not satisfied, for some finite $p_0 \subset p$. But then if $\theta(x)$ is the conjection of p_0, then the negation of θ follows from F, hence is true of a, a contradiction. Thus we have shown that F remains finitely satisfiable if x is replaced by a; iterating this we can solve F.

(2) This is a special case of (1)

(3) Let F_0 be the collection of basic formulas over A. For any $\phi(x)$ let $\phi*(x, y)$ be the formula $\phi(x) \sim \phi(y)$. Then $\{phi*(x, y) : \phi \in F_0\}$ implies that x,y are conjugate by $Aut(U/A)$, and hence implies $\theta*(x)$, where $\theta(x)$ defines X. The result follows.

Definition 3 (Induced structure) • *Let M be an L-structure, and let A be a substructure of M. The induced structure on A is the collection of sets $A^k \cap Y$, where Y is a basic subset of M^k. (The language is not L, but obtained canonically from L and k.)*

Definition 4 *Let U be a universal domain for L. Let X_i be a basic subset of $U^{n(i)}$, and E_i a basic subset of $U^{2n(i)}$, such that E_i is the graph of an equivalence relation on X_i. Let Y_i be the quotient X_i/E_i. Let $\pi_i : X_i \to Y_i$ be the projection. A subset S of $Y_1 \times \ldots Y_m \times X$ is called basic if $\{(x_1, ..., x_m, x) : (\pi_1 x_1, \ldots, \pi_m x_m, x) \in S\}$ is basic. Write $U*$ for U enriched by the new sorts Y_i, and the new basic sets. If the X_i, E_i enumerate all the pairs as above, we write $U* = U^{eq}$.*

Definition 5 *Let U be a universal domain.*

- *We say that $Th(U)$ is stable if every Δ-formula is a stable formula in the sense of Shelah.*
- *We define the (ordinal) Morley dimension and degree of basic subsets of U. Assume the notion of Morley dimension β has been defined, for $\beta < \alpha$, and let X be basic. We say that X has Morley dimension α if it does not have Morley dimension β for $\beta < \alpha$, and if for some m, whenever $X = X_1 \cup \ldots X_{m+1}$, and the union is disjoint, then at least one X_i has Morley dimension $< \beta$. The least such m is the Morley degree of X.*
- *U is Morley if, as a basic subset of U, U has some Morley dimension.*

Lemma 1.7 *1. U is Morley iff there are countably many complete, consistent types in any countable set of formulas of $L(U)$*

2. If U is Morley, then every basic subset of U^n has Morley dimension.

Proof

1. cf. [Sh2]

2. When the language is countable, (1) is equivalent to: countably many conjugacy types in U over any countable set. It follows easily that there are only countably many conjugacy types in U^n over any countable set. In general, one can reduce to this case by finding countable languages such that the condition of 1.2 holds (and working in a universal domain for that language.)

□

Lemma 1.8 *Let U be a universal domain for L, and suppose U has Morley dimension. Then U^{eq} is a universal domain (for the appropriate language.)*

Proof (Compare [PPo].) The compactness property for U^{eq} follows easily from the compactness property for U. To prove homogeneity, we require the following lemma and corollaries.

Lemma 1.9 *Let E be a basic equivalence relation on the basic set X. For $Y \subset X$, let $EY = \{x \in X : (x,y) \in E forsomey \in Y\}$. If Y_1,Y_2 are basic subsets of X, and $EY_1 \subset Y_2$, then there exists a basic Z, $EZ = Z$, and $EY_1 \subset Z \subset Y_2$.*

Proof Suppose not. Since EY_1 is disjoint from $E(X \setminus Y_2)$, the two Σ_1 sets can be separated by a basic set $Y_{3/2}$. Then $EY_1 \subset Y_{3/2}$ and $EY_{3/2} \subset Y_2$; and there is no basic Z with $EZ = Z$ between either Y_1 and $Y_{3/2}$, or between $Y_{3/2}$ and Y_2. In this way we define Y_α for every diadic rational α in the interval. We obtain a strictly increasing dense chain of inclusions; by considering the sets $Y_\beta \setminus Y_\alpha$ we obtain a contradiction to Morley-ness.

Corollary 1.10 *If Y is a basic subset of X, then EY is the intersection of basic E-saturated sets Z_i (i.e.$EZ_i = Z_i$).*

Proof We have $EY = \cap_i W_i$,with W_i basic. By the lemma, there exists $Z_i = EZ_i$ with $EY \subset Z_i \subset W_i$. Clearly $Y = \cap_i Z_i$.

Corollary 1.11 *If X is an E-saturated Σ_1 subset of Z, then EX is the intersection of basic E-saturated sets.*

Proof We have $X = \cap_i Y_i$,with W_i basic, so $X = \cap_i EY_i$, and we can apply the previous corollary.
Proof of 1.8 let A,B be small subsets of U^eq and let $\sigma : A \to B$ be an isomorphism. Let a be an element of U; we will extend σ so that the domain includes a. By compactness, it suffices to show that if b is a tuple from A, and X a basic set with $(a,b) \in X$, then there exists a' with $(a', \sigma(b)) \in X$. Say $b \in Y/E$. Then $T = \{y \in Y : (\exists x)((x,(y/E)) \in X)\}$ is an E-saturated Σ_1-subset of Y, so by 1.11 it is the union of basic E-saturated sets. Hence T/E is the union of basic sets, so it is preserved by σ. Since $b \in T/E$, $\sigma(b) \in T/E$, which is what we needed to show. In this way σ may be extended so that the domain has the following property: whenever Y is basic, E a basic equivalence relation on Y, and $c/E \in A$, then for some $c' \in A$, cEc'. At this point the homogeneity of U can be used to extend $\sigma|U$ to an automorphism of U; this automorphism extends uniquely to U^eq, and necessarily extends σ.

Lemma 1.12 *Any Σ_1-function f is basic*

Proof $(x,y) \notin f$ iff $(\exists y')(y \neq y'$ *and* $(x,y') \in f)$, so the complement of f is also Σ_1, and by separation f is basic.

Lemma 1.13 *Let U be a universal domain.*

1. *If $c \in U$, then the expansion (U, c) is a universal domain for $L(c)$.*

2. *If Y is a basic subset of U^n, then Y, with the induced structure, is a universal domain for $Th(Y)$.*

3. *If Y is a small intersection of basic subsets of U^n, then Y, with the induced structure, is a universal domain for $Th(Y)$*

Proof (1) Homogeneity for U implies homgeneity for (U, c); compactness for (U, c) follows from compactness for U and homogeneity for U.
(3) Homogeneity for X follows from homogeneity for U; note that every automorphism of U preserves X. Compactness for X is also clear from compactness for U.

Lemma 1.14 *Let $\mathtt{U}$ be a universal domain. If $\mathtt{U}$ is a saturated model of $\mathtt{U}$ in the usual first-order sense, or just if $\mathtt{U}$ is $|L|^+$-compact for universal formulas, then the full first order theory of $Th(\mathtt{U})$ is model complete*

Proof Any existential formula is equivalent to a conjunction of basic ones. (Namely, $(\exists x)\phi(x, y)$ to the conjunction of all negations of basic formulas $\sigma(y)$ implying $\neg\phi(x, y)$.) If compactness holds for universal formulas, then any existential formula is equivalent to a basic one.

Proposition 1.2 (Stability) *Suppose U is a universal domain, $Th(U)$ stable.*

1. *(Definability.) Every Δ- type over U is definable, by a Δ-formula.*

2. *(Existence of nonforking extension.) Let $C \subset U$, p a Δ-type over C. Then there exists a basic type $p*$ over G extending p, with $p*$ definable over acl(C).*

3. *(Symmetry.) Let $C = acl(C) \subset U$. Suppose $p(x)$,$q(y)$ are basic types over U, definable over $acl(C)$. Let $\phi(x, y)$ be an arbitrary formula. Let a,b realize $p|C$,$q|C$ respectively. Then $\phi(a, y) \in q$ iff $\phi(x, b) \in p$.*

4. *(Uniqueness.) Let C be algebraically closed in the following sense: the classes of a C-basic equivalence relation with finitely many classes are all C-basic. Then the nonforking extension p^* in (2) is unique. Hence it is defined over C.*

5. *(Forking characterization) Let $p(x)$ be a basic type over $C = acl(C)$, $p*$ its non-forking extension, and suppose $\phi(x, b) \in p*$. Then there exist conjugates b_i of b $(i \in \omega)$ such that $p(x) \cup \not\phi(x, b_i) : i \in \omega\}$ is inconsistent.*

6. *(Finiteness and conjugacy). (Finitely many Γ-types, for finite $\Gamma \subset \Delta$, definable over C and consistent with a given Δ-type over C.)*

Proof The results on local stability citeHP apply, since every basic, or even Σ_1, formula is stable. In the third clause we use the fact that a if $\phi(x, y)$ is an arbitrary formula true of (a, b), it follows from some Δ- formula true of (a, b).

Remark 1.15 *Suppose U is as in the above proposition. Let X be a C-basic set with Morley dimension. Elements of X that are not in any C-basic set of smaller dimension are called generic. There are finitely many types over $acl(C)$ of generic elements. If $\phi(x, y)$ is a Δ-formula over C, there exists a C-atomic $\theta(x)$ such that $\theta(a)$ holds iff for any element b of X, generic over $C \cup a$, $\phi(a, b)$. We express $\theta(x)$ as: "for a generic y in X, $\phi(x, y)$". Using the forking characterization (6), $\theta(a)$ fails iff $\{y \in X : \phi(a, y)\}$ has Morley dimension less than that of X.*

Lemma 1.16 *Let U be a universal domain with Morley dimension. Let P_0 be basic, $M \subset P_0^3$ basic, $P_n \subset P$ basic, $G = \cap_n P_n$. Suppose $M \cap G^3$ defines a group structure on G. Then $G = \cap_n Q_n$ where $(Q_n, M \cap Q_n^3)$ is a group for each n.*

Proof We may assume the Morley dimension and multiplicity of P_n is constant, and that $P_{n+1}P_{n+1} \subset P_n$. (For $x, y \in P_0$, we write xy for the unique $z \in P_0$ such that $(x, y, z) \in M$, if it exists.) It follows that for $a \in G$, $aP_n \cap P_n$ has the same Morley dimension and degree as P_n. Fix $n \geq 2$, and let

$$Q = \{x \in P_1 : \text{ for generic } y \in P_n, xy \in P_n\}$$

Then Q is basic. We have $G \subset Q \subset P_{n-1}$. Further Q is a group under multiplication: if $x, y \in Q$, let $c \in P_n$ be generic to (x, y); then $yc \in P_n$, and by a dimension argument, yc is generic to x; so $xyc \in P_n$.

2 The compact Lascar group

A remarkable connection between compact groups and first order theories was discovered by Lascar [Lascar]. He associated to each first order theory a certain quotient of the automorphism group of a saturated model. For a large class of theories, (the "G-compact" theories,) he showed that this quotient has the structure of a compact topological group.

We will repeat here Lascar's ideas in a slightly more elementary way (particularly with regard to the definition of the topology), influenced by work of Kim and Pillay. We will obtain a compact group associated canonically to any first order theory; for G-compact theories, it is the same as Lascar's group. We will denote Lascar's full group as $LS(T)$ (but we will never use it), while the compact quotient that we work with (and will construct directly) will be denoted $Ls(T)$. Whether $LS = Ls$ in general remains open.

We obtain the group as a group of homeomorphisms of a natural compact topological space associated with T, that we will call the Kim-Pillay space. This space has the type space, and the strong-type space, as quotients, but in general is bigger and has nontrivial connected components. We will call the associated group the compact Lascar group. Examples of Poizat involving actions of real algebraic groups show that any compact real algebraic group can be a Lascar group of some theory.

Recent work of Kim and Pillay [K1], [KP] has shown that *simple* theories are G-compact, and that the Lascar group described here agrees with the one in [Lascar]. The question of the existence or construction of non G-compact theories appears not to have been investigated.

2.1 The Kim-Pillay space

Let $\mathtt{U}$ be an $\aleph_0$-saturated structure. A relation on $\mathtt{U}$ is said to be 0-definable if it is determined by a formula without parameters. If a binary relation on $\mathtt{U}$ is the intersection of 0-definable relations, and is an equivalence relation with at most $2^{|T|}$ classes, we call it a Kim-Pillay relation. The intersection of all Kim-Pillay relations is Kim-Pillay; we denote it by E_{KP}.

Equivalently:

Definition 6 *Let $\mathcal{F}$ be a maximal family of 0-definable reflexive, symmetric binary relations with the following properties:*

1. *If $R \in \mathcal{F}$, every R-anticlique is finite. (An R-anticlique is a set of elements containing no pair from R.)*
2. *If $R \in \mathcal{F}$, there exists $R' \in \mathcal{F}$ such that $R'(x,y) \& R'(y,z) \Rightarrow R(x,z)$.*

In any sufficiently saturated model, E_{KP} is defined by the family $\mathcal{F}$.

To justify the definition, note that the union of any collection of such families again has the same properties.

Note that if $R \in \mathcal{F}$ then for each k there exists R' satisfying (1), and such that if $R'(x_i, x_{i+1})$ for $i = 0, ..., k$ then $R(x_0, x_k)$.

Question 7 *Let L be a countable language. Is the set of pairs (ϕ, T) with $\phi \in \mathcal{F}(T)$, Borel?*

A first step would be to construct a graph of diameter > 2, a finite bound on the size of antichains, and having a 1-point Kim-Pillay space (or to show that this is impossible.)

For our purposes, a *universal domain* for T is an $(2^{2^{|T|}})^+$-saturated, $\aleph_1$-homogeneous model of T. We will speak as if the theory is 1-sorted, though if it is many sorted the same results will apply sort-wise.

Definition 8 *Let T be any first-order theory, or Robinson theory. Let $\mathtt{U}$ be a universal domain for T. Define the Kim-Pillay space of T*

$$X_{KP}(T) = \mathtt{U}\ /E_{KP}$$

We topologize X_{KP} as follows: a basic closed set is the image in X of a definable subset of $\mathtt{U}$ *(possibly with parameters). We also give it a (finitary) structure, where the relations are the images of the 0-definable relations on* $\mathtt{U}$ *.*

If R is a definable n-ary relation, let $\bar{R} \subset {X_{KP}}^n$ be the image of R. It is of course quite possible that $\bar{R} \cap S \neq (\bar{R} \cap \bar{S})$. However, $\bar{R} \cup S = \bar{R} \cup \bar{S}$, so the complements of the basic closed sets form a basis for a topology.

Proposition 2.1 *1. X_{KP} is a compact Hausdorff space.*

2. The finitary relations $\bar{R}$ are closed.

3. If L is countable, X_{KP} has a countable basis for the topology (hence is a Polish space.) Indeed if M is any model, the set of complements of sets $\bar{R}$ with R M-definable, forms a basis for the topology.

4. If $f : X_{KP} \to X_{KP}$ respects the relations $\bar{R}$, then f is 1-1 and onto; and there exists an automorphism of $\mathtt{U}$ *inducing f on X_{KP}.*

Proof 1) For compactness, it suffices to check that every family of basic closed sets C_i with the finite intersection property, has non-empty intersection. Note that the number of closed sets C_i is at most $2^{|X_{KP}|}$ (to give a trivial estimate.) C_i is image modulo E_{KP} of a definable set D_i. The family of the sets D_i has the finite intersection property. By compactness, the D_i have non-empty intersection, hence so do the C_i.

To see that the space is Hausdorff, let c, d be E_{KP}-inequivalent points of $\mathtt{U}$. Since E_{KP} is an equivalence relation, there is no e with (c, e) and (e, d) both in E_{KP}. Using compactness, one can find a 0-definable R, $E_{KP} \subset R$, such that

$$\neg \exists x R(c, x) \& R(x, d))$$

Let $F = \{x : \neg R(c, x)\}$, $F' = \{y : R(c, x)\}$ Then $F \cup F' = \mathtt{U}$. Let $\bar{F}$,$\bar{F}'$,$\bar{d}$, $\bar{c}$ be the images of of F,F'.d,c modulo E_{KP}. Then $\bar{F} \cup \bar{F}' = X_{KP}$. Moreover, $\bar{d} \notin \bar{F}'$, and $\bar{c} \notin \bar{F}$. The complements of $\bar{F}$,$\bar{F}'$ are open sets separating $\bar{c}$,$\bar{d}$. Thus X_{KP} is Hausdorff.

2) Note first that the unary relations are closed by definition. define $X_{KP}(D) = D/(E_{KP}(D))$ in a similar manner for any 0-definable $D \subset \mathtt{U}^{\,m}$ (using the induced structure.) Then the natural map

$$X_{KP}(D_1 \times D_2) \to X_{KP}(D_1) \times X_{KP}(D_2)$$

is surjective and continuous. Since both spaces are compact Hausdorff, the image of a closed set is closed.

3) Let X' be the same set X_{KP}, but topologized by taking only images of M-definable closed sets as basic closed sets. Then the identity $X \to X'$ is continous; in particular X' is compact. We show it is Hausdorff by improving the above argument slightly.

First find a 0-definable R, $E_{KP} \subset R$, such that there are no c', d' with $R(c, c')$,$R(c', d')$ and $R(d', d)$.

Pick a 0-definable R' such that $E_{KP} \subset R' \subset R$, and such that if $(x, y), (y, z) \in R'$ then $(x, z) \in R$. Since E_{KP} has boundedly many classes, every R'-anticlique is finite. Let I be a maximal R'-antichain among elements of M. Then I is finite; since M is a model, I is a maximal R'-antichain in $\mathtt{U}$. Thus there exists $c' \in M$ with $R'(c', c)$.

Let $F = \{x : \neg R(c', x)\}$, $F' = \{y : R(c', x)\}$ Then $\bar{F} \cup \bar{F'} = X_{KP}$. Moreover, $\bar{d} \notin \bar{F'}$, and $\bar{c} \notin \bar{F}$. (If $\bar{c} \in \bar{F}$, then there exists c'' with $c'' E_{KP} c$, and $\neg R(c', c'')$. However $R'(c', c)$ and $R'(c, c'')$, contradiction. If $\bar{d} \in \bar{F'}$, then there exists $d' E_{KP} d$ with $R(c', d')$. But also $R(c, c')$ and $R(d', d)$, a contradiction.) The complements of $\bar{F}$,$\bar{F'}$ are open sets separating c, d. Thus X' is Hausdorff. It follows that the identity map is a homeomorphism, so $X = X'$, as claimed.

4) Let $f : X_{KP} \to X_{KP}$ be a function, preserving the relations. We must find an automorphism s of $\mathtt{U}$ inducing f. Pick some set A of size continuum, such that the image of A under E_{KP} is all of X_{KP}. Then the requirement is:

$$s(a)/E_{KP} = f(a/E_{KP})$$

for each $a \in A$.

By compactness, it suffices to show this can be achieved for any finite subset of A at a time. Let $\{a_1, \ldots, a_n\} \subset A$.

Pick c_i with $c_i/E_{KP} = f(a_i/E_{KP})$. We must merely find $\{b_1, \ldots, b_n\}$ such that $b_i E_{KP} c_i$, and $tp(b_1 \ldots b_n) = tp(a_1 \ldots a_n)$.

If this is impossible, then by compactness, there exists a 0-definable relation C with $(a_1, \ldots, a_n) \in C$, such that there are no $b_i E_{KP} c_i$ with $(b_1, \ldots, b_n) \in C$.

However, since the finitary structure on X is preserved by f, one can lift $f(a_1/E_{KP}, \ldots, a_n/E_{KP})$ to some $b_1, \ldots, b_n$ with $(b_1, \ldots, b_n) \in C$.

This contradiction finishes the proof.

Here is the relation between X_{KP} and the space of strong types, X_{Sh}.

Lemma 2.1 *There exists a canonical map from X_{KP} to the space of strong types. It is continous, and surjective, with connected fibers. In fact the fibers are precisely the connected components of X_{KP}.*

Proof We let E_{Sh} be the intersection of all 0-definable equivalence relations; then E_{KP} refines E_{Sh}, and the map in question sends an E_{KP}-class to the E_{Sh} -class containing it. Surjectivity and continuity are clear. If Y is a connected subset of X_{KP}, the image is a connected subset of X_{Sh}, thus a point. Conversely, if Y is a disconnected closed subset of X_{KP}, let Y_1, Y_2 be complementary closed subsets of Y. Then there exist two infinitely-definable subsets C_1, C_2 of $\mathtt{U}$, whose images modulo E_{KP} are Y_1, Y_2. Since C_1, C_2 have no E_{KP}- equivalent elements, by compactness there exist definable $D \supseteq Y$, $D_1 \supseteq C_1$, $D_2 \supseteq C_2$, $R \supseteq E_{KP}$, such that $R(x,y) \ \wedge \ D_1(x) \ \wedge \ D_2(y)$ is inconsistent, and $D \Rightarrow D_1 \ \vee \ D_2$. (With D,R but not necessarily D_1, D_2 defined over $\emptyset$.) It follows that $D(y) \ \wedge \ D_1(x) \ \wedge \ R(x,y) \Rightarrow D_1(y)$. Thus the equivalence relation E' generated by $R(x,y)$ inside $D(x)$ has more than one class. On the other hand E', being coarser than E_{KP}, has only finitely many classes, so it must be generated by R in finitely manys steps. Thus E' is a definable equivalence relation with finitely many classes. This proves that Y contains more than one strong type. $\square$

2.2 The compact Lascar group

Definition 9 *Let T be any first-order theory (or more generally, a Robinson theory.) Let* $\mathtt{U}$ *be a universal domain for T. The* compact Lascar *group of T is the image of the natural map*

$$Aut(\mathtt{U}\ ) \to Homeo(X_{KP})$$

or equivalently the automorphism group of the structure X_{KP}. It is topologized using the Tychonoff topology of pointwise convergence.

The above actually defines the compact Lascar group of a single sort of T. The compact Lascar group of another sort, even of the sort of k-tuples, may be bigger. One should perhaps call our group the unary compact Lascar group associated with the given sort, and define the full compact Lascar group as the projective limit over all sorts of T^{eq} of their unary Lascar groups. For our considerations this will not really matter and we will stick with the given sort.

Note that the image of $Aut(\mathtt{U}\ )$ in the Tychonoff product X^X is closed. Generally speaking, when f_n are automorphisms, and $f_n \to f$ pointwise, f need not be 1-1 or onto. But it is at least a function $X \to X$, and preserves whatever finitary structure the f_n preserve. By (4) of the Proposition, it follows in our case that f is 1-1, onto, and induced by an automorphism.

Lemma 2.2 *The compact Lascar group L for can equivalently be topologized by the compact-open topology. In other words, if $C \subset X$ is compact and $U \subset X$ is open, then*

$$\{f \in L : f(C) \subset U\}$$

is an open subset of L

Proof An equivalent statement is that if F, F' are closed and $f(F) \cap F' = \emptyset$, then there exists a neighborhood of f in L with the same property. Lift f to an automorphism σ of $\mathtt{U}$. $f(F)$, F' are each an intersection of basic closed sets; thus the intersection of all basic closed sets containing either $f(F)$ or F' is empty; so there exist definable sets C, C' with F, F' contained in the basic closed sets $\bar{C}$,$\bar{C'}$ respectively, and such that $f(\bar{C}) \cap \bar{C'} = \emptyset$. Clearly it suffices to find a neighborhood of f with the same property.

The definable sets C, C' are not only disjoint, but have no E_{KP}- equivalent elements. Thus for some 0-definable R with $E_{KP} \subset R$, $\sigma(C)$ is disjoint from

$$F_2 = \{x : (\exists y)(\exists z)(x, y) \in R, (y, z) \in R, z \in C'\}$$

Let $\{c_1, \ldots, c_l\}$ be a maximal R-anticlique contained in C. Let

$$U = \{g \in L : g(\bar{c_i}) \notin \bar{C'},\ i = 1, \ldots, l\}$$

Then U is an open neighborhood of f in L. If $g \in U$, say g is induced by an automorphism τ. Let $c \in C$. Then $R(c, c_i)$ for some $i \leq l$. So $R(\tau(c), \tau(c_i))$. If also $R(\tau(c), c')$ holds for some $c \in C$, $c' \in C'$, then $\tau(c_i) \in F_2$. But this contradicts the disjointness of $\tau(C)$ from F_2. Thus $\tau(C)$ and C' have no equivalent elements. So $g(\bar{C}) \cap \bar{C'}$ are disjoint. This holds throughout the neighborhood U, proving the lemma. □

Lemma 2.3 *Let $X = X_{KP}$, F the space of functions from X to X, with the Tychonoff topology. The image L of $Aut(\mathtt{U}\ )$ in F is closed. In the induced topology, mutliplication and inversion are continuous. The action of L on X is continuous.*

Proof The fact that the image of L is closed was noted earlier, as a consequence of the proposition (it can also be shown directly using finite anticliques directly.) The rest is elementary topology: To see that multiplication is continuous, let $f, g \in L$, $fg(p) \in U$, with $p \in X$, U an open subset of X. Let C be a compact neighborhood of $g(p)$, contained in $f^{-1}(U)$. Let $O_g = \{g' : g'(p) \in C\}$. Let $O_f = \{f' : f'(C) \subset U\}$. Then $O_f \times O_g$ is a neighborhood of (f, g), whose image under multiplication lies in $\{h : h(p) \in H\}$. We used here the equivalence of the Tychonoff and compact-open topologies on L, and the compactness (hence, local compactness) of X.
The continuity of the action is proved similarly.
Finally, consider inversion. let $f_n \to f$ in L. Let $g_n = {f_n}^{-1}$. Refining the net, using compactness of L, we may assume $g_n \to g$. Then $f_n g_n = g_n f_n = 1$ so by continuity of multiplication, $fg = gf = 1$. Thus ${f_n}^{-1} \to f^{-1}$ as required.

Corollary 2.4 *$Ls(T)$ is a compact topological group.*

Some further results on the compact Lascar group will be proved in §3 and §4.

3 Spaces of Robinson theories

We relate in this section the Kim-Pillay space of a theory to the space $\mathcal{R}$ of all Robinson theories. The natural setting turns out to be not quite topological spaces, but rather certain sequences of topological spaces (needed in order to take account of n-tuples at the same time as elements.) We set up the framework before beginning. We will then show that a certain equivariant form of the Kim-Pillay space embeds into $\mathcal{R}$. Conversely, any space embedding appropriately into $\mathcal{R}$ (relative to a given group action) can be the Kim-Pillay space of some Robinson theory.

We will assume in this section that the Robinson Δ is the family of quantifier-free formulas of the language L. Since we are working with structures varying along a topological space, it would be more natural to let Δ as well as T vary continuously. This appear to involve no real difficulties, but since only the simpler case is required in our applications we will restrict attention to it.

3.1 f-spaces and homogeneous spaces

Definition 10 *An f-space X is a contravariant functor from the category of finite sets into topological spaces. An f-space will be said to be compact, Hausdorff, etc. if each $X(n)$ is. Similarly for maps between f-spaces.*

Writing $n = \{0, \ldots, n-1\}$, we see that such a functor is equivalent to giving a sequence of topological spaces $X(n)$, together with some continous maps $X(n) \to X(m)$ (indexed by maps $n \to m$, and obeying the natural commutation laws.) A map between f -spaces X, Y is a sequence of continuous maps $X_n \to Y_n$, commuting with the maps between them.
The topological spaces we will consider will all be compact, but not all Hausdorff. By a topological group however we always mean a Hausdorff one.

Example 3.1 *Let G be a compact group, acting on a compact topological space X. An f-space X_G can be defined by letting $X_G(n) = X^n/G$ (the quotient space of X^n by G, acting diagonally.) Given $j : n \to m$, $X_G(j) : X_G(n) \to X_G(m)$ is the map induced by way of the natural maps $X^n \to X^m$.*

X_G contains all the "G-equivariant " information in the space X. As we will not need this, let us just note:

Lemma 3.2 *Assume some finite sequence from X has trivial stabilizer. Then the homeomorphism type of X, and of the pair X,G, can be recovered from X_G.*

Proof Let $a \in X^n$ have trivial stabilizer. Let $\bar{a}$ be the image of a in X^n/G.
Let $\pi_n : X_G(n+1) \to X_G(n)$ be the map corresponding to the inclusion of n in $n+1$. Define a map $f : X \to X_G(n+1)$ by $f(x) = \overline{(a, x)}$. Then clearly f is a homeomorphism $X \to {\pi_n}^{-1}(\bar{a})$.
Define a partial ordering on the elements of $\cup_n X_G(n)$: $a < b$ if $a \in X_G(n)$, $b \in X_G(m)$, $n < m$, and $a = (X_G(h))(b)$ for some injective $h : n \to m$. This is a directed system. Clearly what applied to $\bar{a}$ will be true of any larger element. Thus $X \simeq {\pi_m}^{-1}(c)$ for any sufficiently large c (with $c \in X_G(m)$.) This shows how to recover X. The G-conjugacy relation E_m on X^m is now also easily recovered. Finally G itself can be recovered as the group of homeomorphisms of X, preserving each E_m-class for each m. □
Let us include here one more definition.

Definition 11 (diagonal map) *Let X be an f-space. Let $\delta_m : X[1] \to X[m]$ be the map corresponding to the constant function $m \to 1$. More generally, let $\delta_m : X[n] \to X[n \times m]$ be the map corresponding to the projection $n \times m \to n$. These are diagonal maps; $\delta_m(x)$ should be thought of as $(x, x, \ldots, x)$.*

3.2 Robinson spaces

Definition 12 *Let L be a single-sorted relational language. We let $\mathcal{R} = \mathcal{R}(L)$ be the space of all (universal) Robinson theories in a language L. Topologize $\mathcal{R}$ by letting a basic open neighborhood have the form:*

$$\{\tau : A \in \tau\}$$

where A is a universal sentence.

Like the Zariski spaces of algebraic geometry, this space is compact T1 but not Hausdorff; whenever $\tau \subset \tau' \in \mathcal{R}(L)$, we have a specialization of points

$$\tau' \to \tau$$

meaning that τ lies in the closure of $\{\tau'\}$. In particular, in this situation, there exists a continous map $[0,1] \to \{\tau, \tau'\}$, with 0 mapping to the special point τ, and all other $t \in [0,1]$ mapping to τ'. However we will see that $\mathcal{R}$ has interesting Hausdorff subspaces.

We need to move to the f-space framework. We could introduce new sorts S_n (of n-tuples), and let $X(n)$ be the space of all Robinson theories in the language of the new sort S_n. We prefer a more parsimonious approach: instead of including $\mathcal{R}$ as one space in a sequence, we deconstruct $\mathcal{R}$ into a sequence of smaller spaces.

We let $L[n]$ be the language appropriate for describing n subsets of an L-structure, i.e. L together with n new unary predicates $P_1, \ldots, P_n$. We view the P_i as sorts; so they may enter into formulas only via quantifiers, $(\forall x \in P_i)$. But each (r-place) relation of L is interpreted on each r-tuple of sorts.

Let us call a universal sentence of the form

$$(\forall x_1 \in P_1) \ldots (\forall x_n \in P_n)\theta$$

simple. We do not insist that all P_i should occur, but none should occur more than once.

We let $\mathcal{R}[n]$ be the set of all sets of simple universal $L[n]$- sentences, that extend to a Robinson theory in $L[n]$. The topology is generated by the basic open sets $G(\sigma) = \{S : \sigma \in S\}$.

We have natural maps $\mathcal{R}[n] \to \mathcal{R}[m]$ corresponding to maps $h : m \to n$.

Lemma 3.3 *$\mathcal{R}$ is a compact f-space*

3.3 Embedding Kim-Pillay in Robinson spaces

Let T be a Robinson theory in L, and let X be the Kim-Pillay space, and Ls the compact Lascar group. Let $\mathtt{U}$ be a universal domain for T, and $\theta : \mathtt{U} \to X$ the structure map. For $x \in X$, $\theta^{-1}(x)$ is an infinitely-definable subset of $\mathtt{U}$ (with parameters.) Given $x = (x_1, \ldots, x_n) \in X^n$, let $P_i = \theta^{-1}(x_i)$. Let $\mathtt{U}_x = \cup_i P_i$, with the induced L-structure. Then $\mathtt{U}_x$ is itself a universal domain.

We define a map $\tau_T : X \to \mathcal{R}[n]$ by letting $\tau_T(x)$ be the simple universal theory of $\mathtt{U}_x$ in $L \cup \{P_1, \ldots, P_n\}$. The Lascar group acts by automorphisms, hence $\tau_T(x) = \tau_T(gx)$ if $g \in Ls$ acts diagonally. Thus τ_T induces a map $X_{Ls} \to \mathcal{R}$.

Let us say that a universal sentence σ holds above a set $Y \subset X$, if σ belongs to $\tau_T(x)$ for any $x \in Y$.

Proposition 3.1 *Let T be a Robinson theory in L, with Lascar group Ls and Kim-Pillay space X. Then τ_T embeds X_{Ls} homeomorphically into $\mathcal{R}$.*

Proof

Let us prove first the continuity of $\tau_T : X^n \to \mathcal{R}[n]$.

Let U be a basic open subset of $\mathcal{R}[n]$, corresponding to a universal sentence σ. Let $a = (a_1, \ldots, a_n) \in {\tau_T}^{-1}(U)$. We can write σ in the form: $(\forall x_1 \in P_1) \ldots (\forall x_n \in P_n)\beta$, where β is an L-formula. Here each x_i may denote several variables. By compactness, there exist definable $R_1, \ldots, R_n$ such that every element of the E_{KP}-class of a_i satisfies R_i, and $R_1(x_1) \wedge \ldots \wedge R_n(x_n) \Rightarrow \beta$. Now $\theta\neg R_i$ is a closed subset of X^n. The complement G_i of this set contains a_i. And $G_1 \times \ldots G_n \subset \tau_T^{-1}(U)$. This shows that $\tau_T^{-1}(U)$ is open, and that $\tau_T[n]$ is continuous.

Clearly τ_T is Ls-equivariant, $\tau_T(gx) = \tau_T(x)$ for $g \in Ls$. We now show that τ_T induces an injective map on X^n/Ls, with Hausdorff image. Let $E = (E_1, \ldots, E_n)$ and $E' = (E'_1, \ldots, E'_n)$ be two n-tuples of E_{KP}-classes. We must show that if E,E' are not Ls-conjugate, then $\tau_T(E), \tau_T(E')$ are distinct and can be separated by open subsets of $\mathcal{R}$. If E, E' are not conjugate, then $\Pi_i E_i$, $\Pi_i E'_i$ have no conjugate elements. By compactness, there exists a 0-definable n-ary relation D separating $\Pi_i E_i$, $\Pi_i E'_i$. So $\tau_T(E)$ contains the sentence: $(\forall x_1 \in P_1) \ldots (\forall x_n \in P_n)(D(x))$, while $\tau_T(E')$ contains the universal sentence: $(\forall x_1 \in P_1) \ldots (\forall x_n \in P_n)(\neg D(x))$. Thus the image is Hausdorff.

Now injective maps on compact spaces, with Hausdorff images, are homeomorphisms. □

Remark In 3.1, T can be recovered from the image Z of X_{Ls} in $\mathcal{R}$, as follows: $(\forall x_1) \ldots (\forall x_n)\sigma \in T$ iff for every $\tau \in Z[n]$, $((\forall x_1 \in P_1) \ldots (\forall x_n \in P_n)\sigma) \in \tau$

Let $f : X \to \mathcal{R}$ be a map of f-spaces, with image Z. we define a Robinson theory $T(f;x)$ in L as follows. Define $T(f;x)$ to be the set of universal sentences $(\forall x_1) \ldots (\forall x_m)\sigma \in T$ such that $(\forall x_1 \in P_1) \ldots (\forall x_m \in P_m)\sigma \in f(\delta_m(x))$. A similar definition applies when $x \in X[n]$, giving an $L[n]$- Robinson theory $T(f;x)$.

Lemma 3.4 *$T(f;x)$ is a Robinson theory. The theories $T(f;x)$, $T(f;i^*x)$ are compatible when $x \in X^n$, $i : m \to n$.*

Proof Essentially by definition. The point is that the interpolation property 1.1 can be verified locally.

3.4 Hausdorff subspaces as Kim-Pillay spaces

We have thus described a correspondence from theories with a given Kim-Pillay space X and Lascar group Ls, to maps of f-spaces $X_{Ls} \to \mathcal{R}$. We would now like to reverse this correspondence, and ask: given a compact space X , a compact group G acting on X, and a map $f : X_G \to \mathcal{R}$, does there exist a theory T such that $X_{KP}(T) = X$, $Ls(T) = G$, and $\tau_T = f\Gamma$

We first represent K as the automorphism group of a topological structure on X. We impose a finitary structure on X, as follows. A closed subset V of X^n is called *regular* if V is the closure of the interior of V. Each $p \in X^n$ has a basis of regular closed neighborhoods. If U is regular closed, then so is KU. We pick some basis for the topology consisting of regular open sets; and let $L = L(K, X)$ be the family of sets KU, for U in this basis. Thus L is a family of K-invariant regular closed subsets of X.
The language $L = L(K, X)$ will thus have an n-ary relation for each such regular closed $V \subset X^n$. The interpretation of L on X is the tautological one.

Lemma 3.5 *The automorphism group of X as a structure is precisely K. Two tuples from X with the same L-quantifier-free type are K-conjugate.*

Proof K clearly acts by automorphisms. Converesly, let f be an automorphism or partial automorphism of the finitary structure. Note that for each tuple $t \in X^n$, f preserves the set Kt. This is because $\{t\} = \cap t \in V \in LV$, so using compactness, $Kt = \cap t \in V \in LKV$, and each KV is preserved by f.
Given a finite subset S of X, there exists $g_S \in K$ such that $g_S|S = f|S$. By compactness of K, there exists $g \in K$ such that $g_S \to g$ (where the set of finite subsets of X is viewed as a net.) Thus for any $p \in X$, by continuity of the action, $g_S(p) \to g(p)$. But $g_S(p) \to f(p)$. So $g = f$. □

Proposition 3.2 *Let X be a compact Hausdorff topological space, G a group acting on X. Let $f : X_G \to \mathcal{R}$ be a map of f-spaces. Assume:*
(*) If $a \neq b \in X_G[n]$, then there exists a quantifier - free $\sigma(x_1, \ldots, x_n)$ with $((\forall x)\sigma) \in f(a)$, $((\forall x)\neg\sigma) \in f(b)$.
(**) For $a \in X_G[1]$, $f(a)$ is a theory with 1-point Kim-Pillay space.
Then there exists a Robinson theory T with $X_{KP}(T) = X$, $Ls(T) = G$, and $\tau_T = f$

Remark 1 (*) implies:
(*') For each n, $f[n] : X_G[n] \to \mathcal{R}[n]$ is a homeomorphism onto the image.
Indeed it follows from (*) that the map is injective, and that the image is Hausdorff; equivalently, $f : X_G \to R$ is a homeomorphism onto the image.

Remark 2 If (**) is omitted, we still obtain a Robinson theory T, a surjective map $Ls(T) \to G$, and a surjective equivariant map $X_{KP}(T) \to X$. However it is no longer injective; the inverse image of $a \in X$ is homeomorphic to the Kim-Pillay space of $f(a)$.

Proof of 3.2. We first construct an L-structure $\mathbb{U}$ and a map $\theta : \mathbb{U} \to X$. This is equivalent to giving an L_1-structure, where L_1 is a language with a sort S_x for each $x \in X$, and the same k-ary relations between any k-tuple of sorts as L has.

We can define an L_1-structure $\mathbb{U}_1$ as follows. For each $x = (x_1, \ldots, x_n) \in X^n$, let $\mathbb{U}_1(x) = (\theta^{-1}(x_1), \ldots, \theta^{-1}(x_n))$, the interpretation of $S_x = (S_{x_1} \times \ldots \times S_{x_n})$. We demand that $\mathbb{U}_1$ be a universal domain, and that $\mathbb{U}_1(x)$ be a model of $T(f; x/G)$. 3.4 ensures the required consistency. This determines $\mathbb{U}_1$, hence $\mathbb{U}$ and $\theta : \mathbb{U} \to X$.

Next we check that $\mathbb{U}$ is a universal domain as an L-structure.

Lemma 3.6 $\mathbb{U}$ *is compact*

Proof Let A be a small subset of $\mathbb{U}$, and let q be a finitely satisfiable quantifier-free L-type over A. (We do not assume however that q is finitely satisfiable in any single sort of $\mathbb{U}_1$.) Then q can be realized in an ultrapower $\mathbb{U}^*$ of $\mathbb{U}$, say by b. One can view $\mathbb{U}^*$ as the one-sorted restriction of a nonstandard extension of $(\mathbb{U}_1^*, K^*)$ of $(\mathbb{U}_1, K)$. Coalesce the sorts of $\mathbb{U}_1^*$ to those of $\mathbb{U}_1$ using the standard part map $st : K^* \to K$. In other words, define $\theta' : \mathbb{U}_1^* \to X$ by $\theta' = st \circ \theta$. In this way we can think of $\mathbb{U}_1^*$ as an L_1 -structure, rather than an L_1^*-structure.

The continuity of the maps $f[n] : X^n/G \to \mathcal{R}[n]$ implies that when a universal axiom is true in $\mathbb{U}(x)$, it is also true in some neighborhood of x, and hence in the nonstandard monad of x. (I.e. it is true in $\mathbb{U}(x')$ for any x' infinitesimally near x.) It follows that $\mathbb{U}_1^*(x) \models T(f; x/G)$.

Now $\mathbb{U}_1$ is a universal domain; so $\mathbb{U}_1^*$ embeds into $\mathbb{U}_1$ as an L_1-structure, over A. Thus q is realized in $\mathbb{U}$. □

For the proof of homogeneity, note first that the action of any $g \in G$ on X may be lifted to an automorphism of $\mathbb{U}$. The assumption (*) states that given two non-G-conjugate n-tuples $\bar{a}, \bar{a'} \in X^n$, there exists a quantifier-free σ with $(\forall x)\sigma$,$(\forall x)\neg\sigma$ holding in $\mathbb{U}_{\bar{a}}$,$\mathbb{U}_{\bar{a'}}$ respectively. It follows that any two n-tuples from $\mathbb{U}$ with the same basic type, lie above G-conjugate elements of X^n.

Lemma 3.7 $\mathbb{U}$ *is homogeneous*

Proof Let (a_i), (b_i) be sequences realizing the same basic type. Then the θ-images $\bar{a}_i$, $\bar{(b_i)}$ realize the same atomic type in the structure X. Indeed, $R(\bar{a})$ holds in X iff $R'(a)$ holds in $\mathbb{U}$, for each R' with $R \subset int(R')$. Using compactness and 3.5, there exists $g \in K$, $g\bar{a}_i = \bar{b}_i$. Now $(\mathbb{U}_1, g \circ \theta)$ (the sorts have been renamed) is a saturated model of the same universal theory as $(\mathbb{U}_1, \theta)$. Thus there exists an isomorphism between these two structures. So we can extend g to an automorphism (γ, g) of the structure/language pair $(\mathbb{U}_1, L)$. Now the sequences $\gamma(a_i)$,b_i realize the same quantifier-free type in $\mathbb{U}_1$. By the homogeneity of $\mathbb{U}_1$, there exists an automorphism α with $\alpha\gamma(a_i) = b_i$. The composition $\alpha\gamma$ is an automorphism of $\mathbb{U}$ and shows the homogeneity.

□

Lemma 3.8 *The kernel of θ is infinitely-definable in* $\mathtt{U}$ *, and indeed equals E_{KP}.*

Proof The kernel of θ is the intersection of all binary $R \in L$ containing the diagonal of X (or equivalently, containing one point (p,p) on the diagonal.) Thus the Kim-Pillay equivalence relation is at least as fine as $\mathcal{E}$; it cannot be finer, since $\mathtt{U}_1$ has trivial Kim-Pillay space on each sort.

Lemma 3.9 *The Kim-Pillay space is X; the compact Lascar group is K.*

Proof Immediate (for the second part, use 3.5).

Lemma 3.10 *If* $\mathtt{U}_1$ *is simple, so is* $\mathtt{U}$

Proof It was shown above that the sorts of $\mathtt{U}_1$ are the classes of a bounded ∞-definable equivalence relation $\mathcal{E}$ on $\mathtt{U}$. Simplicity can be checked inside each class, with parameters permitted also outside the class. (The infinite indiscernible sequences involved must lie entirely within a single class.) So saturation of $\mathtt{U}$ and simplicity of $\mathtt{U}_1$ imply simplicity of $\mathtt{U}$. □

3.5 A criterion for elementarity

We wish to study the effect on truth of sentences of the construction considered above.

The saturation/coalescence construction By this name (almost as long as the description) let us refer to the formation of $\theta' : U' \to X$, where $\theta^* : \mathtt{U}^* \to X^*$ is the saturation of the situation, $st : X^* \to X$ is the standard part map, and $\theta' = st \circ \theta^*$.

The situation is analogous to the Banach space model theory of Krivine and Henson, cf. [Henson]. There too there is a model and a map into a compact space. However the appropriate construction there is "saturation/reduction" and not "saturation/coalescence". In other words, the construction considered here does nothing to change the truth of quantifier-free formulas, while in the Banach space situation unequal elements can become equal.

There is also a connection to sheaf-theoretic forcing, cf. [?].

Let X be a compact space, U a first-order structure, and let $\theta : U \to X$ be a map. Assume the θ-images of the basic relations are closed. Let $U_p = \theta^{-1}(p)$. Let U_1 be U viewed as a many-sorted structure, with sorts U_p. The corresponding language L_1 has quantifiers $(\exists_p)$,$(\forall_p)$ ranging over U_p.

We can also describe "stable quantification near p". Note that the stable truth of a universal sentence implies the pointwise truth, but for general sentences neither this nor the reverse implications are valid.

Consider a sentence ψ in prenex form:

$$(\forall_{p_1} x)(\exists_{q_1} y) \ldots (\forall_{p_k} x)(\exists_{q_k} y)\theta$$

Then we say that ψ is stably true if for any neighborhood H_k of q_k, there exists a neighborhood G_k of p_k such that ... for any neighborhood H_1 of q_1, there exists a neighborhood G_1 of p_1, with:

$$(\forall_{G_1} x)(\exists_{H_1} y) \ldots (\forall_{G_k} x)(\exists_{H_k} y)\theta$$

where $(\forall_G)$ means: for each $p' \in G$, $(\forall_{p'})$, and dually for $(\exists_G)$.
Let us say that a sentence of L_1 in is stably true if it has a logically equivalent prenex sentence that is stably true.

Lemma 3.11 *Let σ be a prenex sentence of L_1, stably true in U_1. Then σ remains true in $\mathtt{U}\ '$.*

Proof By induction on the innermost quantifier, show this for sentences with parameters from U_1^*.

The following lemma assumes in effect that X is a quotient of the Kim-Pillay space of a structure U. We do not assume that U is saturated; but do assume that the map is surjective, and that U is saturated "pointwise". With an additional stability assumption, we conclude that U is saturated.

Lemma 3.12 *Let U be a structure, $\theta : U \to X$ a map onto a compact Hausdorff space. Assume the image of a basic relation is closed, and that conversely the inverse image of the diagonal $\Delta \subset U^2$ is an intersection of basic, 0-definable, binary relations. Form a many-sorted structure U_1 whose sorts are the fibers of θ. Assume U_1 is saturated. Further assume that the theory of U_1 is axiomatizable by stably true sentences. Then U is saturated.*

Proof Let U^* be a saturated model of the theory $\hat{T}$ of U, of the same cardinality as U. We will show $U \simeq U^*$. Let U' be obtained from U^* as in the saturation - coalescence construction. It suffices to show that U', U_1 are isomorphic. By 3.11,$U' \models Th(U_1)$. Moreover, U' is saturated. Indeed let Φ be a small, finitely-satisfiable collection of unary L_1-formulas. They all refer to some sort U_p, $p \in X$. Now pick $a \in U$ with $\theta(a) = p$. Let $\mathcal{F}$ be a family of 0-definable basic binary relations (closed under finite intersections) whose intersection is the kernel of θ. By compactness of X, for any neighborhood G of p in X, there exists $R \in \mathcal{F}$ such that $R(a)$ is contained in $\theta^{-1}(G)$, and contains $\theta^{-1}(p)$. Let Ψ be the collection of formulas $R(a, x)$, $R \in \mathcal{F}$. So $\Phi \cup \Psi$ is finitely satisfiable in $\mathtt{U}\ ^*$, hence has a solution. This solution must lie in $\mathtt{U}\ _{p'}$ for some p' infinitesimally close to p. So in $\mathtt{U}\ '$, Φ has a solution. Thus $\mathtt{U}\ '$ is a saturated model of $\tilde{T}_1$, so it is isomorphic to $\mathtt{U}\ _1$. □

Proposition 3.3 *Let $\mathtt{U}$ be a universal domain for a Robinson theory T in L, and let $\theta : \mathtt{U} \to X = X_{KP}(T)$ be the Kim-Pillay map. Assume the many-sorted Robinson structure $\mathtt{U}\ _1$ is is first-order, i.e. the universal theory T_1 of $\mathtt{U}\ _1$ has a model completion $\tilde{T}_1$. Further assume that $\tilde{T}_1$ has stably true axioms. Then T is first-order.*

Proof Immediate from the lemma and from 1.14.

The assumption that $\mathtt{U}_1$ is first order is artificial, and can be removed as follows. In general a Robinson theory is axiomatized by universal axioms, together with infinitary axioms of the form:

$$(\forall x)(\exists y) \bigwedge_i \phi_i(x) \Rightarrow \theta(x,y)$$

Let us say that $(\forall_p x)(\exists_q y) \bigwedge_{i \in I} \phi_i(x) \Rightarrow \theta(x,y)$ is stably true if for every neighborhood H of q, there exists a neighborhood G of p and a finite $I_0 \subset I$, such that $(\forall_G x)(\exists_H y) \bigwedge_{i \in I_0} \phi_i(x) \Rightarrow \theta(x,y)$.

Lemma 3.13 *With the above definition, 3.3 remains true without the assumption that* $\mathtt{U}_1$ *is first-order*

More clumsily but more directly, we could also state:

Lemma 3.14 *Let assumptions be as in 3.2. Let* $\mathtt{U}(a)$ *be a universal domain for* $T(f;a)$, $a \in X_G[n]$. *Assume:*

(***) Whenever $a \in X_G[n]$, (c,d) *a tuple from* $\mathtt{U}(a)$, *and* $\mathtt{U}(a) \models \phi(c,d)$ *holds* ($\phi \in L[n]$ *quantifier-free), there exists* $\psi(y)$ *such that* $\mathtt{U}(a) \models \psi(d)$, *and a neighborhood* U *of* a *in* $X_G[n]$, *such that for any* $a' \in \mathtt{U}(a')$, *and any* $d' \in \mathtt{U}(a')$, *there exists* $c' \in \mathtt{U}$ *with* $\phi(c',d')$

Then is T *is first-order, i.e.* T *has a model completion* $\tilde{T}$ *and* $\mathtt{U}$ *is a saturated model of* $\tilde{T}$.

The proof is similar.

4 Simple constructions

4.1 Simple rank-one

We show here that any compact group can occur as the Lascar group of a supersimple Robinson structure of rank one. If the group is separable, the language can be taken to be countable.

Definition 13 *Let* U *be a universal domain (for a Robinson theory). We say* U *is* supersimple of rank 1 *if there is no definable family of definable sets* $F(a)$, *and infinite set of parameters* a_i, *such that* $F(a_i)$ *is infinite for each* a_i, *but* $F(a_i) \cap F(a_j)$ *is finite for* $a_i \neq a_j$.

Remark 4.1 *If* T *is a first order theory of rank one, the independence theorem holds, hence the Lascar group is totally disconnected.*

Proof This follows from Buechler's much more general result, [Bu]. (Another proof, with the same assumptions, was independently found by Shami.) It is also a special case of a result that supersimple theories of bounded weight have connected Lascar group; cf. [W2], Lemma 4.6. More directly related to the motivation for the construction in this section is an older proof, holding in finite rank when an additional definability condition is imposed on the rank. See [ChaH] for a proof in this context, involving the existence of stable formulas. The definability condition is easily seen to be redundant in rank 1:

Lemma 4.2 *Let* $\mathbb{U}$ *be a universal domain of a rank one simple Robinson theory. Let* $F(a)$ *be any definable family of definable sets. Then the set of a with* $F(a)$ *finite is a definable set. Thus* $\mathbb{U}$ *has S1-rank one.*

Proof Suppose otherwise. Then one has a_n with $F(a_n)$ finite but of unbounded size. By Ramsey's theorem, we may assume that for $l < m < n$, $F(a_l) \cap F(a_m) = F(a_l) \cap F(a_n)$. Thus the sets $F(a_l) \setminus F(a_{l+1})$ are pairwise disjoint; so they must be of bounded size. It follows that with $b_l = (a_l, a_{l+1})$, $G(b_l) = F(a_l) \cap F(a_{l+1})$ is of unbounded size; and the $G(b_l)$ are linearly ordered by inclusion. Let $H(b, b') = G(b') \setminus G(b)$. Then this is a uniformly definable family, containing infinite pairwise disjoint subfamilies of arbitrarily large sets ($H(b_k, b_{2k}), H_(b_{2k}, b_{3k}), \ldots$).) So H contains an infinite subfamily of infinite, pairwise-disjoint sets, a contradiction.

Proposition 4.1 *Let* K *be a compact group,* X *a homogeneous space for* K*. Then* X *can be realized as the Kim-Pillay space of a rank 1 simple universal domain, with Lascar group* K*.*

We put a finitary structure on X, as in 3.5 and the discussion preceding it. Call the language $L = L(K, X)$.

$\mathbb{U}_1$ We first construct a many-sorted universal domain $\mathbb{U}_1$. The language has a sort S_p for each point p of X. For each n-ary relation $R \in L$, and each n-tuple $t = (p_1, \ldots, p_n)$ of sorts, there will be a relation $R_t \subset S_1 \times \ldots \times S_p$.
We will have universal axioms that assert:
T1) $(\forall x_1, \ldots, x_n) \neg R_t(x)$ whenever $t \notin R$.
T2) $(\forall x_1, \ldots, x_n) R_t(x)$ whenever $t \in int(R)$
On any particular finite tuple of sorts, each relation is either empty, or full, or nothing is said of it at all. Nor do any axioms relate any two of the relations, on a given product of sorts. Thus it is clear that (T1),(T2) have a model completion T. Let $\mathbb{U}_1$ be a saturated model of T. Let $\theta : \mathbb{U}_1 \to K$ be the map such that $S_\alpha = \theta^{-1}(\alpha)$.
By Proposition 3.2, we obtain a Robinson structure $\mathbb{U}$ with Kim-Pillay space X and map θ : $\mathbb{U} \to X$, and Lascar group K, such that for $a = (a_1, \ldots, a_n) \in X^n$, $\mathbb{U}_a$ is a universal domain for the model completion of $(T1), (T2)$.

Lemma 4.3 $\mathbb{U}$ *is supersimple of rank 1*

Proof It suffices to show that if $D(x, y)$ is a quantifier-free formula in $k+1$ variables x and $y = (y^1, \ldots, y^k)$, and a_i is an infinite indiscernible sequence (matching the variable y), and $D(x, a_i)$ is infinite for one i, then $D(x, a_i) \cap D(x, a_j)$ is infinite for all i, j.
Indeed, by indiscernibility, the coordinates of the a_i all lie in the same E-class. Using the definition of $\mathbb{U}$, it is clear that if $D(c, a_i)$ holds, then there exist infinitely many $c' E c$ such that $tp(c', a_j) = tp(c', a_i) = tp(c, a_i)$.

4.2 First order simple theories

4.2.1 Real valued dimensions

We will use a construction originally intended to build strongly minimal sets, and later $\aleph_0$-categorical stable structures. (cf. [H1], [W1].) In both of these instances, the construction involved (implicitly or explicitly) a certain "generic" structure constrained by a dimension theory; followed by a more complicated construction of a homogeneous substructure of the generic substructure, with the required categoricity and stability properties. Here we will use only the first, generic construction.

This generic theory turned up independently in a probabilistic context considered by Shelah and Spencer. It was studied by Baldwin in these connections and in their own right. See [B].

A small technical modification will be permissible here: we will allow homogeneous amalgamation of a structure B over a substructure A only when B (as well as every proper superset of A within B) has *strictly* bigger expected dimension than A does. This will lead to the independence property, but will not harm simplicity. (For the stable $\aleph_0$-categorical construction, the wish to have strict dimension inequalities was one of the causes for using an irrational number in the base of the theory; this irrational led to other complications that we will not need to face here.)

Let a relational language L be given: a set of sorts, and a set of relation symbols on these sorts. We assume given an assignment of weights $w(S)$, $w(R)$ (non-negative real numbers) to each sort S and to each relation R.

Let w denote this collection of data: sorts, relations, weights. We define a primitive dimension function on finite structures A for this language, as follows. Let $d_0(A) = d_0(A; w)$ be the number of points of A, weighted according to their sorts, minus the weighted sum of the occurences of relations;

$$d_0(A; w) = \sum_{S \text{ sort}} w(S)(S \cap A) - \sum_{R\ r\text{-place relation}} w(R)(R \cap A^r)$$

Let $T_0(w)$ be the universal theory that asserts: no structure with negative d_0 - dimension embeds into the model.

Let B be an L-structure, A a substructure, with $B \setminus A$ finite. let

$$d_0(B/A) = \sum_{S \text{ sort}} w(S)(S \cap (B \setminus A) - \sum_{R\ r\text{-place relation}} w(R)(R \cap (B^r \setminus A^r)$$

This is a real number or $-\infty$. In case B is finite, $d_0(B/A) = d_0(B) - d_0(A)$.

If B is any L-structure, A a substructure, and $d_0(B \cup C/A) \geq 0$ for every finite $C \subset B$, let us write $A \leq_w B$.

If $d_0(C/A') > 0$ for every finite $C \subset B$, and every sufficiently large finite $A' \subset A$, we will write $A <_w B$.

If A is a substructure of an L-structure B, there exists a unique smallest $A' \subset B$ with $A \subset A'$ and $A' \leq_w A'$; we denote it $cl_w(A; B)$ or, if the identity of B is clear, just $cl_w(A)$. We let

$d(A;B) = d_0(A')$.

Note that $d(A;B) \leq d_0(A')$. If A and L are countable, so is A'.

There is also a smallest substructure A' containing A with $A' < B$. Let us denote it $cl'_w(A;B)$. The size of A' cannot in general be bounded in terms of $|A|$.

Let $\mathcal{C} = \mathcal{C}(w)$ be the class of finite L structures A with $\emptyset <_w A$.

If A, B, C are models of T_0, A a substructure of both B and C, and $A \leq_w B$, we define the canonical free amalgam $B \otimes_A C$ of B, C over A to be the disjoint union of B and C over A, as L-structures. Then $C \leq_w B \otimes_A C$. Thus $B \otimes_A C \models T_0$.

Remark In the stable case, the canonical free amalgam is the only free amalgam. Here, relations of weight 0 may be present; we would like to call the amalgam free regardless of whether such relations hold between elements of B and C. Similar weight-zero relations will at all events arise by means of quantification.

Definition 14 *Let A, B, C be substructures of M, $A = B \cap C$. Assume $B \cup C \leq_w M$. We say that B, C are in* free amalgamation *over A within M if whenever R is a relation of weight > 0, and $R(c)$ holds for some tuple d from $B \cup C$, then d is entirely from B or from C*

Lemma 4.4 *Let B, C be in free amalgamation over A within some L-structure D whose universe is $B \cup C$. Assume $A \leq_w B$ (resp. $A <_w B$). Then $C \leq_w D$ (resp. $C <_w D$) and and $D \models T_0$.*

The proof is omitted.

We will also consider an extension-by-definition of the language L. Let $\phi(x, y)$ be a conjunction of atomic formulas, in two finite sets of variables x, y. Let A, B be a structure whose elements form a tuple satisfy ϕ. Assume $d_0(B/B') \leq 0$ for each B' with $A \subset B' \subset B$. Let Φ be the family of all formulas with this property. In this situation, let $d((\exists y)\phi(x, y)) = d_0(B)$. The language L_+ will be the same as L, except that the formulas $(\exists y)(\phi(x, y))$ will be treated as atomic, for $\phi \in \Phi$. (They will be underlined in this capacity.) The universal theory T_+ will assert that $\phi(x, y) \Rightarrow \underline{(\exists y)(\phi(x, y))}$ In addition, T_+ will include all universal closures of formulas $\psi \Rightarrow \bigwedge_i \neg \underline{(\exists y)\phi}$, such that $T_0 \vdash \psi \Rightarrow \bigwedge_i \neg \phi$

Definition 15 *Let Ψ be the collection of universal closures of :* $(\exists y)(\phi(x, y)) \iff \underline{(\exists y)(\phi(x, y))}$. *An L_+-structure A is* natural *if $A \models \Psi$.*

Remark If A, B are L-structures, A a substructure of B, A_+, B_+ are the natural L_+- expansions, and $A <_w B$, then A_+ is a substructure of B_+.

Lemma 4.5 *Let $M \models T^+$. Then M is existentially closed iff M is natural, and whenever $A \leq_w M$, $B \in \mathcal{C}$, $A <_w B$, B finite, there exists an embedding $j : B \to M$, $j|A = Id$, with $jB < M$.*

Proof First assume M is existentially closed. If $\underline{(\exists y)(\phi(a, y))}$ holds, the atomic diagram of M together with T must be consistent with $\phi(a, y)$; otherwise for some c from M such that $\phi'(a, c)$,

$T \vdash \neg(\phi' \ \wedge \ \phi)$, so $T_+ \vdash \phi' \Rightarrow \neg(\underline{(\exists y)(\phi(a,y))})$, a contradiction. By existential closure, $\phi(a,y)$ is realized in M. This shows that M is natural. The second property follows from the amalgamation lemma 4.4. One can freely amalgamate B, M over A; obtain a model M' of T_0 with $B \leq M'$ and $M < M'$; and interpret L_+ naturally in M'. Since $M < M'$, M is an L_+-substructure of M'. By saturation and existential closure, B embeds into M over A.

Conversely, suppose M satisfies the above properties. To show that M is existentially closed, we may assume M is saturated. Let M' be a model of T_+, M a substructure of M', $A \subset M$ finite. Let B be a finite subset of M', containing A. We must show that B embeds into M over A, by an L_+-embedding. It suffices to deal with a finite amount of the quantifier-free L_+ -type of B at a time. This type contains some underlined existential formulas and some negations of such. We can handle the underlined existentials by making the corresponding existential formulas true, enlarging B to include witnesses for them. Thus it suffices to find an embedding j preserving the negated (underlined) existential formulas true of B. This will be done be ensuring that the image $jB <_w M$.

We can amalgamate either with respect to $\leq_w$, or to $<_w$. In both cases a model complete theory exists, and is simple, with trivial Lascar group. The language is L in the case of $\leq_w$, L_+ in the case of $<_w$. The model completions are denoted $\tilde{T}$,$\tilde{T_+}$ respectively. We will prove the lemmas only for the case of T_+.

Lemma 4.6 *T_+ has a model completion $\tilde{T_+} = \tilde{T_+}(w)$. $\tilde{T_+}$ is complete.*

Proof Let us show that the class of existentially closed models of T^+ is elementary. Let M be a saturated existentially closed model. The first property in 4.5 is by definition elementary. The second one implies a stronger version of itself, that is obviously elementary:

Suppose $A \leq_w M$, $B \in \mathcal{C}$, $A \leq_w B$, B finite, and $cl'_w(A;B)$ embeds into M. Then this embedding can be continued to an embedding of B in M.

To see this, replace A by $cl'_w(A;B)$ in the original property.

Completeness follows from the joint embedding property of $\mathcal{C}$ (canonical free amalgamation over $\emptyset$.) $\square$

For a formula or partial type ϕ over B, let $d(\phi) = sup\{d(c/B) : \phi(c)\}$. This supremum is actually attained, as one immediately sees either by invoking compactness, or directly. For ϕ a formula (without parameters) in Φ, it agrees with the previous definition. By amalgamation, it does not change if B is enlarged.

Let $\mathtt{U} \ = \mathtt{U} \ (w)$ be a universal domain for $\tilde{T_+}$.

Lemma 4.7 *Let $B \leq_w \mathtt{U}$, $B_i = cl_w(B \cup \{b_i\})$ $(i = 1,2)$. Then B_1, B_2 are in free amalgamation over B iff $d(b_1/B_2) = d(b_1/B)$*

Proof One direction is immediate. For the other suppose B_1, B_2 are not in free amalgamation. So a relation of weight $\alpha > 0$ holds between them. Let ${B_1}'$ be a finite subset of B_1 such that the relation holds between ${B_1}'$ and B_2. Let $B' \subset B$ be finite, such that $d(b_1/B') - d(b_1/B) < \epsilon$. Let ${B_2}' = cl_w(B' \cup \{b_2\})$. Then $d(b_1/B_2) \leq d(b_1/{B_2}') \leq d(b_1/B') - \epsilon < d(b_1/B)$.

Lemma 4.8 *If $p \in S(a)$, $\phi(L(b))$, and $d(p \cup \phi) = d(p)$, then ϕ does not fork over p.*

Proof Let $\rho = d(p)$. Note that if $\phi \Rightarrow \phi_1 \ \vee \ \phi_2$, then $d(p \cup \phi_1) = \rho$ or $d((p \cup \phi_2) = \rho$. Thus it suffices to show that in a model M of $\tilde{T_+}$, there is no indiscernible sequence b_i, $tp(b_i/a) = tp(b/a)$, such that $p \cup \bigwedge_{i \in \omega} \phi(x, b_i)$ is inconsistent for for any k-element set of indices ω.
Suppose otherwise. The b_i can be taken to be independent over some set containing B containing a , i.e. $d(b_i/B) = d(b_i/B \cup \{b_j : j \neq i\})$. (E.g. continue the sequence into negative indices and let $B = a \cup \{b_{-i} : i\}$.)
Let c solve $p \cup \{\phi(x, b)\}$, $d(c/ab) = \rho = d(p)$.
Extend $p \cup \{\phi(x, b)\}$ to a type $q = tp(c/B)$, with $d(q) = \rho$. We have $\rho = d(c/Bb) \leq d(c/B) \leq d(c/a) = \rho$. Thus $cl_w(Bc)$ is in free amalgmation with $cl_w(Bb)$ over B. Let $B_* = cl'_w(Bb; M)$, $B_i = cl'_w(Bb_i; M)$, $C = cl'_w(Bc; M)$, $D = cl'_w(Cb; M)$,
Construct an L-structure E containing M, as well as a copy C' of C, and $c' \in C'$, such that there are isomorphisms $h_i : B \cup C \to E$ over B with $h_i(b) = b_i$, $h_i(B_* = B_i)$, $h_i(c) = c'$, $h_i(C) = C'$; and with no relations between M and C' other than those implied by this. Then by 4.4, $E \models T_0$, and $M <_w E$. Embed E into a model $\tilde{M}$ of $\tilde{T_+}$, in such a way that $E <_w \tilde{M}$. $p \cup \bigwedge_{i \in \omega} \phi(x, b_i)$ is realized in $\tilde{M}$, by c', a contradiction.

Lemma 4.9 *$\tilde{T_+}$ is simple.*

Proof For any a, B , we must show that there exists $B_0 \subset B$ of bounded size, such that $tp(a/B)$ does not fork over B_0. Pick B_0 such that $d(a/B) = d(a/B_0)$, and use 4.8

Lemma 4.10 *$\tilde{T_+}$ has trivial Lascar group*

Proof The independence theorem holds for types over $\emptyset$.

Lemma 4.11 *T_+ depends continously on the weights w, in the following sense. If σ is a universal consequence of $T_+(w)$, then there exist finitely many sorts S_i and relations R_j, and $\epsilon > 0$, such that for any other system of weights w', if $|w(S_i) - w'(S_i)| < \epsilon$ and $|w(R_j) - w'(R_j)| < \epsilon$, then σ is also a consequence of $T_+(w')$*

Proof With underlines removed, the sentence σ is actually a consequence of some axioms of $T_0(w)$. Thus it suffices to show that each of these axioms hold in $T_0(w')$. A typical axiom of $T_0(w)$ asserts that a particular L-structure A, with negative w-weight, does not embed into the model. For sufficiently near w', A will also have negative w'-weight, so the same sentence will be a sentence of $T_0(w')$.

4.3 A rank one simple $\aleph_0$-categorical structure

The construction of the §5.1 can be used to build simple $\aleph_0$-categorical structures, that do not have locally modular geometries.

Let L be a one-sorted language, with sort S, and a n 3^n-ary relations $R_{n,i}$ for each $1 \leq i \leq n$. Let $w(S) = 1$, $w(R_{n,i}) = 1$ for each n.

We consider only L structures where $R_n(a_1, \ldots, a_n)$ implies that $a_i \neq a_j$ (call these irreflexive.) This is to avoid having uncountably many, or even infinitely many, nonisomorphic structures of a given finite size. We will also require symmetry (this for an inessential reason, the "two" in 4.16.)

Let $\mathcal{C}$ be the family of finite irreflexive L-structures, whose every substructure has at least as many points as relations:

$$\mathcal{C} = \{A : |A| < \aleph_0,\ (\forall B \subset A) d_0(B) \geq 0\}$$

Let

$$\mathcal{C}' = \{A \in \mathcal{C} :\ (\forall B \subset A)|B| \leq 3^{d_0(B)-1}\}$$

In particular, $\mathcal{C}'$ contains structures of any size $0, 1, 2, \ldots$ with no relations. The smallest structure in $\mathcal{C}'$ bearing a relation has three elements and one ternary relation, making for dimension 2.

Lemma 4.12 *$\mathcal{C}'$ has the joint embedding property, and the $<$-amalgamation property. Indeed if $A, B, C \in \mathcal{C}'$, $A <_w C$, $A <_w B$, then $B \otimes_A C \in \mathcal{C}'$*

Proof We may assume $|A| \leq |B| \leq |C|$. In addition by the strict inequality, $d_0(A) < d_0(B)$. Thus if $D = B \otimes_A C \in \mathcal{C}'$, we have

$$d_0(D) = d_0(C) + (d_0(B) - d_0(A)) \geq d_0(C) + 1$$

so $3^{d_0(D)-1} \geq 3 \cdot 3^{d_0(C)-1} \geq 3|C|$. Yet $|D| = |C| + (|B| - |A|) \leq 2|C|$. Thus $|D| \leq 3^{d_0(D)-1}$.
Any subset D' of D is itself a free amalgam, $D' = (B \cap D') \otimes_{A \cap D'} (C \cap D')$. So the same argument applies and shows $|D'| \leq 3^{d_0(D')-1}$. Thus $D \in \mathcal{C}'$ $\square$

Thus we may form the amalgamation limit M' of $\mathcal{C}'$. It is a homogeneous substructure of a model $M(w)$ of the theory $\tilde{T_+}(w)$ constructed above. But now for any finite $A \subset M'$, $cl_w(A; M')$ and even $cl'_w(A; M')$ are finite. indeed, the cardinality of $cl'_w(A; M')$ cannot exceed $3^{d(A;M')-1} \leq 3^{d_0(A)-1}$. We will refer to cl'_w simply as closure. So the closure of a finite set is finite.

The type of a closed substructure A of M' is determined by the isomorphism type; hence there are only finitely many types of closed substructures of a given finite size. Any finite substructure of M' embeds into a closed one of bounded dimension, hence also size. Thus:

Lemma 4.13 *$Th(M')$ is $\aleph_0$-categorical. It is homogeneous over algebraically closed subsets.*

Lemma 4.14 (Independence theorem) *Let $E \in \mathcal{C}'$, $B' \in \mathcal{C}'$ a substructure. Let $A_0, A_1, A_2, A_{01}, A_{02}, A_{12}$ be closed substructures of E. Assume $A_{ij} \in \mathcal{C}'$, and $A_{ij} = cl'_w(A_i \cup A_j)$. Assume A_0, A_1, A_2 are free amalgamation over a closed substructure B. Further assume that $E = \cup_{ij} A_{ij}$, and that the relations on E are just the union of the relations on the A_{ij}. Then $E \in \mathcal{C}'$.*

Proof A_{12} is in free amalgamation over $A_1 \cup A_2$ with $A_{01} \cup A_{02}$. Thus at all events, using 4.4 twice, $E \in \mathcal{C}$. Let F be a substructure of E. We must show that $|F| \leq 2^{d_0(F)-1}$. We may assume $F = cl'_w(F;E)$, since taking closure increases size and does not increase d_0. Let $F_i = F \cap A_i$, $F_{ij} = F \cap A_{ij}$. Then F_i, F_{ij} are closed substructures of A_i,A_{ij}. So they are in $\mathcal{C}'$. $F = \cup F_{ij}$. We have $|F_{ij}| = 3^{d_{ij}-1}$, $d_{ij} = d_0(F_{ij})$. Say d_{12} is the largest of the d_{ij}. If $d_{12} = d_0(F)$, then $F \subset cl'_w(F_{12}) = F_{12}$, and the result follows already from 4.12. Otherwise, $|F| \leq 3 \cdot 3^{d_{12}-1} = 3^{d_{12}} \leq 3^{d_0(F)-1}$. □

Remark The independence version inductively implies a stronger version, where $n+1$ independent sets A_i $(i = 0, \ldots, n)$, together with the closures A_{0i}, A_{1i}, and $A_{1,\ldots,n}$.

Lemma 4.15 *M' is simple of rank 1.*

Let $\phi(x,b)$ be a formula in one variable x. If $d(\phi) = 1$, then following the proof of 4.8, using lemma 4.14, ϕ does not fork over $\emptyset$. If $d(\phi) = 0$, then $a \in cl'_w(b;M')$ so we saw above that $a \in acl(b)$.

The following additional facts are clear:

Lemma 4.16 *M' is transitive and primitive. It has precisely two 2-types of distinct points: those whose closure has two points, and those whose closure has three. Thus the algebraic closure relation gives a non-homogeneous matroid (in the sense of Zil'ber.)*

Remark The above is a modification of the construction of stable, $\aleph_0$-categorical structures, (cf. [W1]). The stable case involved an irrational α with poor rational approximations from below. Now if β is rational, it has one good rational approximation, but the ones strictly below it are as poor as for any irrational. The argument is thus actually simplified. The difficulty in the stable case was that the gap between $< \beta$ and $\leq \beta$ creates a region of the structure not controlled by numerical dimension; this is however not a problem if one only wishes for simplicity.

The relation to the rank of the theory is this: using rational α as a weight for a single relation, on a sort of weight 1, yields a superstable theory, of rank ω^a. For irrational α, the theory is a limit of such, stable but unranked. Using $\alpha = 1$, even with strict inequalities, retains rank 1.

Remark Simplicity is associated with the independence theorem; the property that $P(3)^-$ -diagrams can be completed. This was obtained cheaply by letting the growth rate of algebaic closure be exponential with base 3. I did not check, but assume the generalized independence theorem fails in M'. (This states that all $P(n)^-$-diagrams can be completed; see the definition below.) However, if 3^{d-1} is replaced by $(d+1)!$, the resulting structure will have the $P(n)^-$-amalgamation property for all n.

We give here a somewhat weak version of the amalgamation properties:

Definition 16 *The $P(n)^-$-amalgamation property is the following:*

Let $I = P(n) \setminus \{n\}$, ordered by inclusion. Let $(M_s, j_{s,t})$ be a directed system of substructures of $\mathtt{U}$ *, with index set I. Assume:*

(1) For any $s \in P(n)^-$, $\{j_{i,s}M_{\{}i\} : i \in s\}$ is independent over $j_{\emptyset,s}M_\emptyset$.

(2) $M_s = \mathrm{acl}(\cup_{t\in s} j_{t,s}(M_{\{}t\}$.

Then the directed system extends to one on $P(n)$, with (1) valid for $s = n$.

This leads to the following question. Let M be an infinite combinatorial geometry, perhaps carrying additional structure. Assume:

1. For some constants C, b, every set of rank k has at most Cb^k elements.
2. Any isomorphism between closed subsets of M, extends to an automorphism of M.
3. The generalized independence theorem holds.

(Note that (1),(2) implies that M is $\aleph_0$-categorical, while (3) for implies simplicity.)
Must M be locally modularΓ

5 Appendix: Amalgamation and the saturation spectrum

We answer here a question from [Sh1].

Let $\lambda = \lambda^{|T|} \geq \kappa$. Consider the property:

$SP_T(\lambda, \kappa)$: Every model of T of power λ extends to a κ-saturated model of power λ.
Shelah shows in [Sh1],[Sh4]

1. T non-simple implies $SP_T = SP_\infty =_{def} \{(\lambda, \kappa) : \lambda = \lambda^{<\kappa}\}$.
2. T stable implies $SP_T(\lambda, \kappa) = SP_1 =_{def} \{\text{all } \lambda, \kappa\}$.
3. λ strong limit, T unstable implies $SP_T(\lambda, \kappa)$ iff $\lambda = \lambda^{<\kappa}$.
4. Let $SP_2 =_{def} SP_T$ where T is the theory of the random graph. Then (Engelkind–Karlowitz) $\mu = \mu^{<\kappa}$, $\mu \leq \lambda \leq 2\mu$ implies $(\lambda, \kappa) \in SP_2$.
5. $SP_2 \subseteq SP_T$ for any simple unstable T.
6. In some model of ZFC: for all simple unstable T, $SP_T = SP_2 \neq SP_\infty$).
7. In some model of ZFC: for some T with a simple predicate P, $SP_{T,P} \neq SP_2$).

Question: Does the statement in (6) follow from ZFCΓ]

Definition. Let $2 \leq k < m$. By a *k-graph* we will mean a structure (A, R), R a k-ary relation on A, such that $R(a_1, \ldots, a_k)$ implies that $a_1, \ldots, a_k$ are distinct, and that $R(a_{f1}, \ldots, a_{fk})$ holds for any permutation f. Thus we will sometimes consider R as a collection of k-sets. An *m-clique* is a subset of A whose every subset of size k is in R. A k-graph is *m-free* if it contains no m-clique. The *generic m-free k-graph* is the unique countable m-free k-graph embedding every finite m-free

k-graph, and admitting quantifier elimination. $T_{k,m}$ is the theory of this graph. $T_k = T_{k,k+1}$. We will also consider the generic $(k+1)$-partite $(k+1)$-free k-graph: it is divided into $k+1$ sorts, any k-edge consists of points from distinct sorts, there is no $k+1$ clique, and otherwise anything can happen. The theory of this will be denoted T'_k.

Lemma 1. *$T_{2,m}$ is not simple. For $k > 3$, $T_{k,m}$ is simple unstable, and in fact has Shelah degree 1. Similarly T'_k.*

Proof. Let Γ be a model of $T_{k,m}$. Let $(a(i)(i \in \omega))$ be an indiscernible sequence of n-tuples $a(i,1), \ldots, a(i,n)$, $A_i = \{a(i,1), \ldots, a(i,n)\}$, and let $D(x, a(i))$ be a complete atomic type with infinitely many solutions. We must show that $\{D(x,a(i)) : i\}$ is consistent. Define a k-graph Δ consisting of $\cup_i A_i \cup \{c\}$, where c is a new point; and with $R(b)$ if true in Γ, or if $c \in b$, $b - \{c\} \subseteq A_i$ for some i, and $D(x, a(i))$ so dictates. By indiscernibility, c solves $D(x, a(i))$ consistently for each i. It remains only to check that Δ is m-free. Suppose B is an m-clique. By definition of Δ, $c \in B$, and every $(k-1)$-subset of $B - \{c\}$ is in A_i for some i. Since the A_i's form a Δ-system and $(k-1) \geq 2$, it follows that for at most one i is $B - \{c\} - A_i$ nonempty. Hence $B - \{c\} \subseteq A_i$ for some i and so B as a k-graph is described by D, so it is not an m-clique.

Lemma 1.1. *Let $k \geq 3$. Then $SP(T_{k,m}) \subseteq SP(T'_k)$.*

Proof. In each case it is more convenient to think of the universal part of the theory. Use the following interpretation: given an m-free k-graph Γ, let $\Gamma_1, \ldots, \Gamma_k$ be copies of Γ, and $\Gamma_{k+1} = [\Gamma]^{m-k}$; and let a be a k-edge iff a is a subset of $\cup_i \Gamma_i$ containing one point from each Γ_i except for $i = i_0$, and either a is a k-edge of Γ and $i_0 = k+1$, or $\cup a$ is an $(m-1)$-clique of Γ and $i_0 < (k+1)$. This gives a $(k+1)$-free $(k+1)$-partite k-graph Γ'. This is an interpretation of the universal theories, in the sense that every model of T'_k embeds into one of the form Γ', Γ a model of $SP(T_{k,m})$. From this the lemma follows.

Proposition 2. *It is consistent with ZFC that $(\aleph_\omega, \aleph_1) \notin SP(T'_k)$, while $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_\omega$, so $(\aleph_\omega, \aleph_1) \in SP_2$.*

Let Γ be a k-partite k-graph. By $(\Gamma)^1$ we will mean the 1-subsets of Γ with each point from a different sort. Say that Γ is κ-representable if there exists a map $\xi : (\Gamma)^{k-1} \to P(\kappa)$ such that for $u \in (\Gamma)^k$, u is a k-edge of Γ iff $\cap\{\chi(v) : v \in [u]^{k-1}\} = \emptyset$.

Lemma 3. *If $(\aleph_\omega, \aleph_1) \in SP(T'_k)$ then every k-partite k-graph on $\aleph_\omega$ is $< \aleph_\omega$-representable.*

Proof. Let Γ be a k-partite k-graph on $\aleph_\omega$ and is not $\aleph_n$-representable for any n. Let $\Gamma' = \omega \times \Gamma$ be the model-theoretic disjoint union of ω copies Γ_n of Γ (sortwise; with no new edges). View Γ as being a model of the universal part of T'_k, with one sort, say S, being empty. Let Γ^* be an $\aleph_1$-saturated model of T'_k containing Γ and of size $\aleph_\omega$; let M be the interpretation of the sort S in Γ^*, and let $M = \cup_n M_n$, $\mathrm{card}(M_n) = \aleph_n$. Define $\chi_n : (\Gamma_\nu)^{k+1} \to P(M_n)$ by letting

$$\chi_n(u) = \{y \in M_n : u \cup \{y\} \text{ is a } k\text{-edge of } \Gamma^*\}.$$

Suppose for contradiction that no χ_n is a representation of Γ_n. Then for each n there exists $b_n \in (\Gamma_n)^k$ such that either (i) b_n is an edge and $\cap\{\chi(v) : v \in [b_n]^{k-1}\} \neq \emptyset$, or (ii) b_n is a non-edge but $\cap\{\chi(v) : v \in [b_n]^{k-1}\} = \emptyset$. Now (i) is impossible since Γ^* is $(k+1)$-free. So (ii) holds. Thus there are no k-edges of Γ' among $\cup_n b_n$; so it is consistent to demand an element $c \in S(\Gamma^*)$ such that $u \cup \{c\}$ is an edge of Γ^* for each $u \in [b_n]^{k-1}$, $n < \omega$. Say $c \in M_n$. Then $c \in \cap\{\chi(v) : v \in [b_n]^{k-1}\}$, contradicting (ii). Thus some Γ_n, equivalently Γ, is $\aleph_n$-representable.

Proof of Proposition 2. We start with a ground model satisfying GCH, and force $\aleph_\omega$ new subsets of $\aleph_1$. Actually it is convenient to directly obtain a k-partite k-graph Γ on $\aleph_\omega$, by forcing a function $F : \aleph_\omega^k \to \{0,1\}$, and considering it as the characteristic function of such a graph Γ (in which $\aleph_\omega$ plays the role of each of the k sorts; formally $\Gamma = (k) \times \aleph_\omega$). A forcing condition is any function $p : D \to \{0,1\}$ with $D \subseteq \aleph_\omega^k$ countable; the partial ordering is inclusion. We will show that Γ is not $< \aleph_\omega$-representable.

The forcing has the $\aleph_2$-chain condition and is $\aleph_1$-closed, hence adds no new countable sequences, collapses or singularizes no cardinals, and makes $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_\omega$.

Suppose $\chi : (\Gamma)^{k-1} \to P(\kappa)$ is a (name for a) representation, $\kappa < \aleph_\omega$, $k \geq \aleph_1$. Let p_0 be a condition that forces this fact. For any $y \in (\Gamma)^{k-1}$, and any $\alpha \in \kappa$, let $I(y,\alpha)$ be a maximal antichain above p_0 deciding whether $\alpha \in \chi(y)$. Given $u \in [\aleph_\omega]^1$, and $t \in [k]^1$, let $t^*u = \{(t_1,u_1),\ldots,(t_1,u_1)\}$ where $t = \{t_1 < \ldots < t_1\}$, $u = \{u_1 < \ldots < u_1\}$. Let

$$J(u) = \cup\{I(t^*u,\alpha) : \alpha \in \kappa, t \in [k]^{k-1}\}$$

$$D(u) = \cup\{\mathrm{dom}(p) : p \in J(u)\} \cup \mathrm{dom}(p_0).$$

By the $\aleph_1$-chain condition, each $I(y,\alpha)$ has size at most $\aleph_1$, and so $J(y)$ and hence $D(y)$ have size at most κ. $\aleph_\omega$ being sufficiently larger than κ, it is possible to find $w \in [\aleph_\omega]^k$ such that:

$$(*) \qquad \text{If } u \in [w]^{k-1},\ w = u \cup \{x\}, \text{ then } x \notin D(u).$$

Let $b = k^*w$. There are two cases.

Case 1. p_0 forces: $\cap\{\chi(v) : v \in [b]^{k-1}\} = \emptyset$. In this case let p_1 be the condition extending p_0 and stating that b is not a k-edge of Γ. By $(*)$, b is disjoint from $\mathrm{dom}(p_0)$, so this is consistent. But then p_1 forces that b is a counterexample to the definition of a representation, a contradiction.

Case 2. Not case 1. Then for some generic filter G containing p_0, and some $\alpha \in \kappa$, in the extension by G we have $\alpha \in \cap\{\chi(v) : v \in [b]^{k-1}\}$. Thus for each $v \in [b]^{k-1}$, there exists $p_v \in G \cap I(y,\alpha)$ and forcing $\alpha \in \chi(v)$. Discarding G now, we keep the information that $\cup\{p_v : v \in [b]^{k-1}\}$ is a condition. But by $(*)$, $b \notin \mathrm{dom}(p_v)$ for any $v \in [b]^{k-1}$. Thus

$$\cup\{p_v : v \in [b]^{k-1}\} \cup \{(b,1)\}$$

is a condition. But this condition forces $\alpha \in \cap\{\chi(v) : v \in [b]^{k-1}\}$, and also that b is an edge. Again a contradiction to the definition of a representation. Thus Γ is not $< \aleph_\omega$-representable in the generic extension, and by Lemma 3 the proposition is proved.

References

[B] Near model completeness and 0-1 laws. Logic and random structures (New Brunswick, NJ, 1995), 1–13, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 33, Amer. Math. Soc., Providence, RI, 1997

[Bu] S. Buechler,

[CH] G. Cherlin and E. Hrushovski, 'Lie coordinatised structures.' In preparation.

[Henson] Ward Henson, article in "Nonstandard Analysis - Recent developments". ed, A.E.Hurd (LNM 983)

[HKS]

[H1] A new strongly minimal set, in *Stability in model theory, III* (Trento, 1991). Ann. Pure Appl. Logic 62 (1993), no. 2, 147–166.

[H2] E. Hrushovski, A stable $\aleph_0$-categorical structure, unpublished notes

[H3] E. Hrushovski, manuscript on PAC structures, unpublished

[HP] Hrushovski, E. & Pillay, A., "Groups Definable in Local Fields and Pseudo-Finite Fields", *Israel Journal of Mathematics* **85** (1994) pp. 203-262

[ChaH] E. Hrushovski, Z. Chatzidakis,

[KP] Byunghan Kim & Anand Pillay; Simple theories, to appear in Annals of Pure and Applied Logic.

[HML] E. H

[K1] B. Kim, 'Forking in simple unstable theories', *J. of London Math. Soc.* To appear.

[KP] B. Kim and A. Pillay, 'Simple theories', *Ann. of Pure and Applied Logic.* To appear.

[Lascar] D. Lascar, 'On the category of models of a complete theory', *J. of Symbolic Logic* 47 (1982) 249-266.

[PPo] A. Pillay and B. Poizat, 'Pas d'imaginaires dans l'infini!' *J. of Symbolic Logic* 52 (1987) 400-403.

[Sh1] S. Shelah, 'Simple unstable theories', *Ann. Math. Logic* 19 (1980) 177-203.

[Sh2] S. Shelah, *Classification theory*, revised (North-Holland, Amsterdam, 1990).

[Sh3] S. Shelah, Towards classifiying unstable theories, Annals of Pure and APplied Logic 80 (1996) 229-255

[Sh4] Saturation over a predicate, Notre Dame J. **22** (1981),239–248

[W1] Wagner, Frank O. Relational structures and dimensions. Automorphisms of first-order structures, 153–180, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994.

[W2] Wagner, Frank O., Groups in Simple Theories, preprint